hep–th/0501127
CERN-PH-TH/2005-004
UO-HET/515
January 2005

Aspects of WZW models at critical level*

Ioannis Bakas†

Theory Division, Department of Physics, CERN
CH-1211 Geneva 23, Switzerland
`ioannis.bakas@cern.ch`

Christos Sourdis

Department of Physics, Graduate School of Science, Osaka University
Toyonaka, Osaka 560-0043, Japan
`sourdis@het.phys.sci.osaka-u.ac.jp`

Abstract

We consider non-compact WZW models at critical level (equal to the dual Coxeter number) as tensionless limits of gravitational backgrounds in string theory. Special emphasis is placed on the Euclidean black hole coset $SL(2,R)_k/U(1)$ when $k = 2$. In this limit gravity decouples in the form of a Liouville field with infinite background charge and the world-sheet symmetry of the model becomes a truncated version of W_∞ without Virasoro generator. This is regarded as manifestation of Langlands duality for the $SL(2,R)_k$ current algebra that relates small with large values of the level in the two extreme limits. However, the physical interpretation of the $SL(2,R)_k/U(1)$ coset model below the self-dual value $k = 3$ remains elusive including the non-conformal theory at $k = 2$.

---

*Based on talks presented by I.B. at the *37th International Symposium Ahrenshoop on the Theory of Elementary Particles*, Berlin, August 23-27, 2004 and by C.S. at the *RTN and EXT Meeting on the Structure of Spacetime*, Kolymbari, September 5-10, 2004; to appear in the proceedings published in Fortschritte der Physik.

†On sabbatical leave from Department of Physics, University of Patras, GR-26500 Patras, Greece; e-mail: bakas@ajax.physics.upatras.gr

Wess-Zumino-Witten (WZW) models based on non-compact groups G_k, and their gauged versions, provide a wide class of exact conformal field theory backgrounds (or building blocks thereof) for studying string propagation in non-trivial space-time geometries. The level k of the underlying current algebra is typically constrained to take continuous values $k > g^\vee$ by unitarity, where $g^\vee$ is the dual Coxeter number given by the value of the quadratic Casimir operator of G in the adjoint representation. The large k limit dictates the semi-classical geometry of these models by associating a metric, an antisymmetric tensor field and a dilaton that satisfy the vanishing condition of the beta function equations to lowest order in the perturbative α' expansion (see, for instance, [1]). Of course, there can be subleading corrections to higher orders in perturbation theory that are computable in closed form by Hamiltonian methods (see, for instance, [2]). They are in accord with the perturbative renormalization of general non-linear sigma models in two dimensions. In this group theoretical setting the parameter α' is supplied by $(k - g^\vee)^{-1}$ and therefore WZW models at critical level $k = g^\vee$ can be regarded as exact tensionless theories, [3, 4]. This limit is ultra-quantum in nature and it is only possible to consider for non-compact groups. Furthermore, it is a singular limit within the class of conformal field theories, since the central charge of the corresponding Virasoro algebra blows up to infinity, but it makes perfect sense in the more general class of two-dimensional quantum field theories.

We are set to examine the nature of WZW models at critical level with emphasis on the simplest case of $SL(2, R)_k$ current algebra for which $g^\vee = 2$. After a brief overview of the main problems and shortcomings of the effective field theory of such limiting models we will resort to world-sheet methods, which are well defined for all values of the level, and use them to explore their symmetries. Higher rank generalizations are also possible to consider using the appropriate technical ingredients. For further details on these and other related issues we refer the reader to our published work, [4].

The simplest (yet interesting) example of this kind is provided by the gauged WZW coset $SL(2, R)_k/U(1)$, which admits the interpretation of a two-dimensional Euclidean black hole in the large k limit, [1],

$$ds^2 \simeq k \left(dr^2 + \tanh^2 r \ d\theta^2\right) , \qquad \Phi \simeq 2\log\left(\cosh r\right) , \tag{1}$$

since the semi-classical geometry looks like a semi-infinite long cigar satisfying the equation $R_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi$ imposed by conformal invariance to lowest order in perturbation theory. Higher order corrections transform the background to the perturbatively exact form, [2],

$$ds^2 \simeq (k - 2) \left(dr^2 + \beta^2(r) \ d\theta^2\right) , \qquad \Phi \simeq \log\left(\frac{\sinh 2r}{\beta(r)}\right) \tag{2}$$

for all values of k, where $\beta(r)$ is given by

$$\beta^{-2}(r) = \coth^2 r - \frac{2}{k} \ . \tag{3}$$

Thus, by pretending that the result is valid all the way down to the critical level, it

readily follows as $k \to 2$ that

$$ds^2 \simeq (k-2)\left(dr^2 + \sinh^2 r \, d\theta^2\right), \qquad \Phi \simeq \log\left(\cosh r\right), \tag{4}$$

which corresponds to the geometry of an infinitely curved hyperboloid in the ultra-quantum regime.

The reasoning of the effective theory should be taken with care when the level of the current algebra becomes critical. It is well known that in all tensionless theories there is an abundance of states that become massless and as such they should be taken on equal footing with the massless modes provided by the metric, dilaton and anti-symmetric tensor field (where it is applicable). For, it is not consistent to truncate the action only to the generic massless modes without accounting for the contribution of the "will be massless" states in the ultra-quantum regime. Otherwise, the effective geometry appears to be singular, as in the case noted above. This is a well known phenomenon which cannot be controlled here by the inclusion of a few extra modes, as in other more tractable cases. As a result, there is no systematic method to study WZW models at critical level, such as $SL(2,R)_2/U(1)$, within the framework of the local effective field theory, which is only suitable for summarizing the results of the perturbative expansion. This shortcome is also consistent with the observation that non-linear sigma models with non-compact target spaces are non-perturbatively non-renormalizable in the sense that there can be divergencies giving rise to counterterms that contain higher dimensional operators and which correspond to coupling of gravitons to massive fields, [5]. These divergencies arise when the (momenta)^2 are equal or larger than $1/\alpha'$, in which case any truncation of the perturbative expansion does not make sense. Hence, their effect becomes dominant in the tensionless limit $\alpha' \to \infty$. All these provide indications that non-commutative geometry might be a better framework to formulate tensionless models by means of infinitely non-commutative structures, but we will not pursue this line of thought further in the present work.

There is an alternative description of the bosonic $SL(2,R)_k/U(1)$ model for small values of k (close to its critical value) in terms of the sine-Liouville conformal field theory, which is roughly

$$S = \frac{1}{4\pi}\int \sqrt{h} d^2x \left(\frac{1}{k-2}(\partial r)^2 + \frac{1}{k}(\partial\theta)^2 - \frac{1}{k-2}R[h]r + \mu^2 e^{-r}\cos\theta\right), \tag{5}$$

where h is the world-sheet metric with Ricci scalar $R[h]$ and μ some parameter, following a conjecture by Fateev, Zamolodchikov and Zamolodchikov (FZZ) (for an overview see, for instance, [6]). The relation between $SL(2,R)_k/U(1)$ and the sine-Liouville models is a strong-weak coupling duality on the world-sheet. The coset conformal field theory becomes weakly coupled in the limit $k \to \infty$, where the cigar geometry is weakly curved. On the other hand, the semi-classical limit of the sine-Liouville theory arises as $k \to 2$, where it becomes weakly coupled, unlike the coset space theory that becomes strongly coupled. Both theories exhibit an asymptotic region $r \to \infty$ where the geometry is that of a cylinder with respective radii $\sqrt{k}$ and $1/\sqrt{k}$ with linear dilaton. The potential term

of the sine-Liouville action is negligible for large r, as it decays exponentially, and the two asymptotic geometries are simply related to each other by T-duality along the angular direction. The non-trivial aspect of the FZZ duality comes from the diminishing size of the S^1 fiber as one approaches the tip of the cigar, which in turn modifies the naive duality rules and it accounts for the generation of the potential term by vortex-instanton effects on the sine-Liouville side using mirror symmetry. Of course, such relation has only been proven rigorously for a supersymmetric variant of these models, namely for the corresponding Kazama-Suzuki coset and $N = 2$ Liouville theory, [7], as originally proposed in [8]. Nevertheless, since the potential term is unbounded from below as r ranges on the whole real axis, it is natural to expect that for small values of k the semi-classical reasoning can be employed to render the sine-Liouville model unstable. This observation is also consistent with the fact that the coset theory is well defined only for $k > 2$.

The limiting value $k = 2$ appears to be rather special for both sides of the FZZ correspondence as the central charge of the Virasoro algebra

$$c = 2\,\frac{k+1}{k-2} \tag{6}$$

becomes infinite. It manifests in the form of a Liouville field with infinite background charge that effectively decouples from the rest of the spectrum leading to a contraction of the Virasoro algebra to an abelian structure. The precise nature of the remnant model is not completely understood to this date but some of its symmetry aspects can be revealed using world-sheet methods. Its proper interpretation in the framework of quantum field theories will also help to explore the gravitational undressing of more general string backgrounds, for which an equivalent description as Liouville theory coupled to other matter fields is only valid asymptotically. Note that the contraction of the Virasoro algebra is also common to other tensionless string models and as a result there is no need to have critical space-time dimensionality for consistent string propagation (see, for instance, [3] and references therein). The meaning of space-time geometry itself is also questionable in the tensionless limit, but there is no better definition of string theory at this moment which can accommodate the notion of tensionless objects in a background independent way.

Ideally, we would like to have a field theoretic manifestation of Langlands (electric-magnetic) duality for $SL(2,R)_k$ current algebra with a level relation (see, for instance, [9])

$$k' - 2 = \frac{1}{k-2}\ . \tag{7}$$

Thus, the limit that k takes its critical value on one side could be compared directly with the large k limit on another side, where the space-time interpretation of the theory is valid. In both cases there are important simplifications that occur and certainly the FZZ correspondence is pointing to the right direction by comparing two different models. However, a drawback of the current formalism is the apparent instability of the sine-Liouville theory for small values of k that might also include $k = 2$. Also, it is not clear

what is the phase of the remnant theory after decoupling of the Liouville field which carries infinite background charge. One might naively think that such decoupling will remove the infinity from the Virasoro central charge and leave the conformal symmetry intact but with finite (possibly zero) central charge. It has been suggested in a different context that $SL(2,R)_2/U(1)$ might be a topological conformal field theory with zero net central charge that arises by taking an average value prescription for c as $k \to 2^{\pm}$, [10]. The subsequent analysis of the world-sheet symmetries of the coset model shows that there is no such remnant of the conformal symmetry and thus its interpretation remains an open question.

We further note here that according to the relation (7) there is a *self-dual* value of the level for

$$k = k' = 3 \ . \tag{8}$$

This value provides the boarder line for the existence of normalizable zero modes of the $SL(2,R)_k/U(1)$ coset associated to marginal operators with dimension 1 (see, for instance, [7] for a complete discussion). For $k > 3$ the existence of this zero mode prohibits changes of the asymptotic radius of the cigar geometry (given by $\sqrt{k}$ in appropriate units), whereas for $k < 3$ its absence seems to allow for such possibility. Clearly, this is a very important difference that holds the key for the physical interpretation of the $SL(2,R)_k/U(1)$ coset for small values of k (including $k = 9/4$ with Virasoro cental charge $c = 26$), [11]; it also shows up in recent attempts to understand the black hole (non)-formation in the singlet sector of matrix models, [12]. Finally note that the $N = 2$ supersymmetric case appears to be more tractable in many respects with the relation (7) being replaced by $k \to 1/k$. Then, the critical level is $k = 0$, while the self-dual value is shifted to $k = 1$. It will also be interesting (and possibly easier) to understand the field theoretic description of the supersymmetric coset at $k = 0$ given its equivalence to $N = 2$ Liouville theory.

We think that better understanding of the Langlands duality for current algebras and its implications for gauged WZW models will help to clarify the situation. In this respect, the ultimate understanding of $SL(2,R)_k/U(1)$ at critical level seems to be lying at the heart of the problem. In the following we restrict attention to the world-sheet symmetries of the coset model $SL(2,R)_k/U(1)$ and examine the structure of its W-algebra in two extreme cases $k = \infty$ and $k = 2$ that correspond to the semi-classical gravitational limit and the tensionless limit, respectively. It will be shown with the aid of non-compact parafermions, and their operator product expansion, that the world-sheet symmetry linearizes in both limits and can be subsequently identified with the W_∞ algebra and its non-conformal higher spin truncation, respectively, [4]. This result can be considered as a special manifestation of Langlands duality, but it is not clear how it generalizes for intermediate values of k. The parafermion currents do not seem to have a definite transformation rule among themselves under (7), which in turn prevents to relate the structure of the corresponding W-algebras for general values of k. After all, given the FZZ correspondence (or mirror symmetry in the $N = 2$ supersymmetric case), the field theoretic realization of such duality is not manifest within the same coset, but

it rather relates different classes of models.

Recall that the lowest parafermions currents of the non-compact WZW model, $\psi_{\pm 1}(z)$, admit the following realization in terms of two-dimensional free fields, [13, 14, 15],

$$\psi_{\pm 1}(z) = \frac{1}{\sqrt{2k}} \left(\mp\sqrt{k-2}\partial\phi_1(z) + i\sqrt{k}\partial\phi_2(z) \right) \exp\left(\pm i \sqrt{\frac{2}{k}}\phi_2(z) \right) \tag{9}$$

for all $k \geq 2$. The fields $\phi_i(z)$ are both space-like with normalized two-point functions

$$< \phi_i(z)\phi_j(w) >= -\delta_{ij}\log(z-w) \ . \tag{10}$$

The operator product expansion of parafermions assumes the form

$$\psi_{+1}(z)\psi_{-1}(w) = \frac{1}{(z-w)^{2\Delta}} \left(1 + \frac{2\Delta}{c_\psi}(z-w)^2 T_\psi(w) + \mathcal{O}(z-w)^3 \right) , \tag{11}$$

where Δ is the conformal dimension of $\psi_{\pm 1}(z)$ and c_ψ is the central charge of the Virasoro algebra with values

$$\Delta = 1 + \frac{1}{k} \ , \qquad c_\psi = 2\frac{k+1}{k-2} \ . \tag{12}$$

The associated stress-energy tensor of the parafermion theory is represented as

$$T_\psi(z) = -\frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 + \frac{1}{\sqrt{2(k-2)}}\partial^2\phi_1 \ , \tag{13}$$

whereas the higher order terms in the expansion (11) give rise to higher conserved chiral currents and their free field realization in terms of $\phi_1(z)$ and $\phi_2(z)$. All these currents generate the extended world-sheet symmetry of the parafermion theory, which is also known as W-algebra.

The parafermions of the coset can be dressed with an additional $U(1)$ field $\chi(z)$, which is assumed to have two-point function $< \chi(z)\chi(w) >= -\log(z-w)$, and the composite fields

$$J^\pm(z) = \sqrt{k}\psi_{\pm 1}(z)\exp\left(\pm\sqrt{\frac{2}{k}}\chi(z) \right) , \qquad J^3(z) = -\sqrt{\frac{k}{2}}\partial\chi(z) \tag{14}$$

satisfy the $SL(2,R)_k$ current algebra with level $k \geq 2$, [14], since

$$\begin{aligned} J^+(z)J^-(w) &= \frac{k}{(z-w)^2} - 2\frac{J^3(w)}{z-w} \ , \\ J^3(z)J^\pm(w) &= \pm\frac{J^\pm(w)}{z-w} \ , \\ J^3(z)J^3(w) &= -\frac{k}{2(z-w)^2} \ , \end{aligned} \tag{15}$$

up to non-singular terms. In this case, the Sugawara construction for the $SL(2,R)_k$ current algebra yields the stress-energy tensor

$$T(z) = \frac{1}{2(k-2)} \left(J^+J^- + J^-J^+ - 2J^3J^3 \right)(z) = -\frac{1}{2}(\partial\chi)^2 + T_\psi(z) \tag{16}$$

with total Virasoro central charge $c = 1 + c_\psi$. At $k = 2$ the center of the enveloping of the current algebra becomes non-trivial, as it is generated by $(J^+J^- + J^-J^+ - 2J^3J^3)(z)$, and it accounts for the contraction of the Virasoro algebra to an abelian structure.

Next, let us examine the parafermion currents in the large k limit. They assume the following simple form,

$$\psi_{+1}(z) = -\frac{1}{\sqrt{2}}\partial\bar{\Phi}(z) \ , \qquad \psi_{-1}(z) = \frac{1}{\sqrt{2}}\partial\Phi(z) \ , \tag{17}$$

where $\Phi(z)$ is a complex free boson taken to be $\Phi = \phi_1 + i\phi_2$. In this case, the parafermions become local abelian currents of dimension $\Delta = 1$ and their operator product expansion (11) can only give rise to boson bilinear terms of the form $\partial^k\bar{\Phi}\partial^{s-k}\Phi(z)$ to each order s in the $z - w$ expansion with $k = 1, \cdots, s-1$. A convenient quasi-primary basis of the W-algebra generators is chosen to be [4, 15]

$$W_s(z) = 2^{s-4}s\frac{(s-2)!}{(2s-3)!!}\sum_{k=1}^{s-1}(-1)^k\binom{s-1}{k}\binom{s-1}{k-1}\partial^k\bar{\Phi}\partial^{s-k}\Phi(z) \tag{18}$$

describing fields for all integer values of spin $s \geq 2$. For $s = 2$, in particular, one recovers the stress-energy tensor $T_\psi(z) = W_2(z)$ of the coset model in the large k limit. The operators $W_s(z)$ generate a linear infinite dimensional algebra which is identified with W_∞.

On the other hand, at the critical level $k = 2$, there is a drastic reduction in the structure of parafermion currents as one of the two free bosons decouples completely. In this case the parafermions assume the form

$$\psi_{+1}(z) = \frac{1}{\sqrt{2}}\partial\bar{\Psi}(z) \ , \qquad \psi_{-1}(z) = -\frac{1}{\sqrt{2}}\partial\Psi(z) \ , \tag{19}$$

where

$$\Psi(z) = e^{-i\phi_2(z)} \ , \qquad \bar{\Psi}(z) = e^{i\phi_2(z)} \tag{20}$$

are fermionic currents that are described as vertex operators of the remaining free boson. Then, since $< \bar{\Psi}(z)\Psi(w) >= 1/(z-w)$, it follows that the operator product expansion (11) gives rise to fermion bilinear terms of the form $\partial^k\bar{\Psi}\partial^{s-k-1}\Psi(z)$ to each order s in the $z-w$ expansion with $k = 1, \cdots, s-2$. However, unlike the previous (semi-classical) case, there is no contribution to order $s = 2$ because the coefficient $2\Delta/c_\psi \to 0$ as $k \to 2$. This is also consistent with the absence of conformal symmetry for WZW models at critical level. As for the remaining generators which arise to higher orders in the expansion, it is convenient to choose the basis, [4],

$$\tilde{W}_s(z) = 2^{s-3}s(s+1)\frac{(s-3)!}{(2s-3)!!}\sum_{k=1}^{s-2}(-1)^{k+1}\binom{s-1}{k+1}\binom{s-1}{k-1}\partial^k\bar{\Psi}\partial^{s-k-1}\Psi(z) \tag{21}$$

for all $s \geq 3$ and identify their algebra with a consistent truncation of W_∞ by moding out the Virasoro algebra.

The above identifications can be made precise by considering the class of infinite dimensional Lie algebras of W_∞ type, [16],

$$[V_n^s,\ V_m^{s'}] \ = \ ((s'-1)n-(s-1)m)\,V_{n+m}^{s+s'-2}+\sum_{r\geq 1}g_{2r}^{ss'}(n,m;\mu)V_{n+m}^{s+s'-2-2r}+ \\ c_s(\mu)n(n^2-1)(n^2-4)\cdots(n^2-(s-1)^2)\delta_{s,s'}\delta_{n+m,0}\ . \tag{22}$$

The commutation relations of the Fourier modes can be cast into operator product expansions for the corresponding currents

$$V_n^s=\oint_0\frac{dz}{2\pi i}z^{n+s-1}V_s(z) \tag{23}$$

and employ contour integration as in two dimensional field theories. In either case, the structure constants of the algebra are determined by the following expressions,

$$g_{2r}^{ss'}(n,m;\mu)=\frac{\phi_{2r}^{ss'}(\mu)}{2(2r+1)!}N_{2r}^{ss'}(n,m)\ , \tag{24}$$

where

$$\phi_{2r}^{ss'}(\mu)=\sum_{k=0}^{r}\frac{(-\frac{1}{2}-2\mu)_k(\frac{3}{2}+2\mu)_k(-r-\frac{1}{2})_k(-r)_k}{k!(-s+\frac{3}{2})_k(-s'+\frac{3}{2})_k(s+s'-2r-\frac{3}{2})_k} \tag{25}$$

and

$$N_{2r}^{ss'}(n,m)=\sum_{k=0}^{2r+1}(-1)^k\binom{2r+1}{k}(2s-2r-2)_k[2s'-k-2]_{2r+1-k}[s-1+n]_{2r+1-k}[s'-1+m]_k\ . \tag{26}$$

Finally, the ascending and descending Pochhammer symbols are defined as usual,

$$(a)_n=a(a+1)\cdots(a+n-1)\ ,\qquad [a]_n=a(a-1)\cdots(a-n+1)\ , \tag{27}$$

whereas the coefficients of the central terms are given collectively by the expression

$$c_s(\mu)=2^{2(s+|\mu|)-7}c\frac{(s+2\mu)!(s-2\mu-2)!}{(2s-1)!!(2s-3)!!}\ . \tag{28}$$

The class of these algebras depends on a parameter μ that can take all values $\mu=-1/2,0,1/2,1,\cdots$. For each such value the operator content of the algebra truncates consistently to all integer spin $s\geq 2\mu+2$, since the structure constants vanish otherwise. Thus, for $\mu=-1/2$ the algebra is $W_{1+\infty}$, for $\mu=0$ it is W_∞, whereas for higher values of μ the resulting algebras do not contain the Virasoro generators nor any other fields with spin less than $2\mu+2$ in their spectrum. Thus, in this context we find that the W-algebra of the $SL(2,R)_k/U(1)$ coset model at $k=\infty$ corresponds to $\mu=0$ with $V_s(z)=W_s(z)$ as given by equation (18) and the (Virasoro) central charge is $c=2$ equal to the dimension of the classical black hole geometry. Likewise, the W-algebra of the coset model at $k=2$ corresponds to $\mu=1/2$ with $V_s(z)=\tilde{W}_s(z)$ as given by equation (21) and the coefficient of the central terms turns out to be $c=2$ using the normalization

given above, [4]. In this case, however, the value of the central charge does not have a direct physical interpretation in the absence of Virasoro generators.

The resulting algebraic structures are very similar to each other following by consistent truncation of the Virasoro generators. Actually, the two limiting cases are formally related at the level of parafermion currents $\psi_{\pm 1}$ by the bose-fermi relation (up to a sign)

$$\Phi(z) \leftrightarrow \Psi(z) \ , \tag{29}$$

as can be readily seen from equations (17) and (19) above. We view this result as manifestation of Langlands duality with the level relation (7) in the two extreme limits of its validity. Then, the form of the corresponding W-generators (18) and (21) simply follows by performing the operator product expansions and choosing appropriate basis to diagonalize the central terms of the algebras. It will be interesting to understand the meaning of this result in a wider context and extend its description beyond the two limiting values $k = 2$ and $k = \infty$ in an appropriate way.

The decoupling of the Liouville field at critical level is one of the most important aspects of WZW models for non-compact groups. Essentially, it rips off one space-time dimension and takes the remnant theory in a non-conformal phase. This disintegration is also seen using other free field realizations of the $SL(2, R)_k$ current algebra, such as the standard Wakimoto representation, [17],

$$\begin{aligned} J^-(z) &= \beta(z) \ , \qquad J^3(z) = \beta\gamma + \sqrt{\frac{k-2}{2}}\partial\varphi(z) \ , \\ J^+(z) &= \beta\gamma^2(z) + \sqrt{2(k-2)}\gamma\partial\varphi(z) + k\partial\gamma(z) \ , \end{aligned} \tag{30}$$

where (β, γ) are commuting ghost fields with dimensions $(1, 0)$ and φ is a space-like boson, all having normalized two-point functions. When $k > 2$ three fields are required to provide a faithful representation of the algebra, whereas for $k = 2$ only two suffice as the contribution of φ to the currents vanishes. The stress-energy tensor following from the Sugawara construction is expressed as

$$T(z) = \beta\partial\gamma - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{\sqrt{2(k-2)}}\partial^2\varphi \tag{31}$$

and therefore φ is a Liouville field with infinite background charge as $k \to 2$. However, after the decoupling the theory is not the ordinary (β, γ) conformal system. Thus, the WZW model and its gauged version are both suffering a drastic change at critical level.

In view of the disappearance of all φ dependence from the bosonic realization (31), a complementary understanding of this essential change may also be provided through the use of restricted Wakimoto modules (see, for instance, [18] and references therein) to investigate the singular structure of the highest weight modules for $k = 2$. For this limiting level value, the Kac-Kazhdan equation degenerates to an identity and the corresponding affine modules contain an infinite number of null states obtained by the application of the commuting - at this limit - Virasoro modes, leading to a vanishing determinant formula,

[19] (but see also [14]). This generic feature of all affine Lie algebras at criticality is in full accordance with the aforementioned decoupling of the Virasoro generator from the spectrum of the W-algebra as the level approaches its critical value.

There is another disintegration limit of the $SL(2,R)_k/U(1)$ coset model that works in the opposite direction and arises even semi-classically. It is based on the observation that when the $U(1)$ gauge group is boosted by an infinite amount the resulting theory is equivalent to the null gauged WZW model $SL(2,R)_k/E(1)$ that describes the dynamics of the Liouville field alone. This reduction of dimensionality can also be reached in a controlled way by considering a very large boost, in which case the semi-classical metric describes $c = 1$ matter coupled to Liouville field acting as a slowly varying background, [20]. In either case, the theory remains conformal all the way up to its disintegration limit for all $k > 2$. At $k = 2$ the null gauged WZW model describes a Liouville field with infinite background charge but there is no other remnant as in the ordinary case. It will be interesting to examine the manifestation of the boosting in the framework of the FZZ correspondence and in particular for the $N = 2$ mirror models for all values of k including the critical level.

The breakdown of the conformal invariance which is seen at critical level might be similar in nature to the Kosterlitz-Thouless transition of a $c = 1$ free boson compactified on a circle. In that case, the compactness of the target space allows for the existence of vortices which are irrelevant at large values of the radius, but they break conformal invariance at small values; in fact the theory is equivalent to the sine-Gordon model. It is conceivable that non-perturbative effects in the $SL(2,R)_k/U(1)$ coset, as those considered in [21], could play a prominent role in this respect. Further support towards this analogy is also provided by the free field realization of the coset model at critical level in terms of a single compact space boson. Starting from the semi-classical regime one may look at the problem by considering the asymptotic radius of the Euclidean black hole which is $\sqrt{k}$ and is shrinking as k is lowered. Although the semi-classical reasoning should not be further trusted due to the perturbative $1/k$ corrections to the background geometry, one is led to suspect that a transition might occur to non-conformal phase for sufficiently small k. The absence of normalizable marginal deformations that already occur for $k \leq 3$ also shows that the physical interpretation of the $SL(2,R)_k/U(1)$ coset should be modified for small values of k, since changes of the radius become possible without affecting the dilaton (unlike $k > 3$); actually, at $k = 2$ all the discrete representations are squeezed out from the spectrum, following [22], thus signaling a transition caused by the liberation of the bound states. Although the FZZ correspondence provides an alternative way to look at this problem its resolution is still lacking and calls for further work.

Acknowledgments

This work was supported in part by the European Research and Training Networks "Superstring Theory" (HPRN-CT-2000-00122), "Quantum Structure of Space-time and the Geometric Nature of Fundamental Interactions" (HPRN-CT-2000-00131) and "Constituents, Fundamental Forces and Symmetries of the Universe" (MRTN-CT-2004-005104). One of us (C.S.) is also thankful to the Japanese Ministry of Education, Culture, Sports, Science and Technology (Monbukagakusho) for financial support during his stay in Osaka.

References

[1] E. Witten, "String theory and black holes", Phys. Rev. D44 (1991) 314.

[2] R. Dijkgraaf, E. Verlinde and H. Verlinde, "String propagation in a black hole geometry", Nucl. Phys. B371 (1992) 269.

[3] U. Lindstrom and M. Zabzine, "Tensionless strings, WZW models at critical level and massless higher spin fields", Phys. Lett. B584 (2004) 178.

[4] I. Bakas and C. Sourdis, "On the tensionless limit of gauged WZW models", JHEP 0406 (2004) 049.

[5] S. Das and B. Sathiapalan, "New infinities in two-dimensional non-linear sigma models and consistent string propagation", Phys. Rev. Lett. 57 (1986) 1511.

[6] V. Kazakov, I. Kostov and D. Kutasov, "A matrix model for the two-dimensional black hole", Nucl. Phys. B622 (2002) 141.

[7] K. Hori and A. Kapustin, "Duality of the fermionic 2d black hole and $N = 2$ Liouville theory as mirror symmetry", JHEP 0108 (2001) 045.

[8] A. Giveon and D. Kutasov, "Little string theory in a double scaling limit", JHEP 9910 (1999) 034.

[9] B. Feigin and E. Frenkel, "Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras", Int. J. Mod. Phys. A7 [Suppl. 1A] (1992) 197.

[10] J. Ellis and N. Mavromatos, "High energy QCD as a topological field theory", Eur. Phys. J. C8 (1999) 91.

[11] J. Maldacena, "Searching for a 2-d black hole", talk delivered at Strings 2004, Paris.

[12] J. Karczmarek, J. Maldacena and A. Strominger, "Black hole non-formation in the matrix model", hep-th/0411174.

[13] J. Lykken, "Finite reducible realizations of the $N = 2$ superconformal algebra", Nucl. Phys. B313 (1989) 473; O. Hernandez, "Feigin-Fuchs bosonization of Lykken parafermions and $SU(1, 1)$ Kac-Moody algebras", Phys. Lett. B233 (1989) 355.

[14] L. Dixon, M. Peskin and J. Lykken, "$N = 2$ superconformal symmetry and $SO(2,1)$ current algebra", Nucl. Phys. B325 (1989) 329.

[15] I. Bakas and E. Kiritsis, "Beyond the large N limit: non-linear W_∞ as symmetry of the $SL(2,R)/U(1)$ coset model", Int. J. Mod. Phys. A7 [Suppl. 1A] (1992) 55.

[16] C. Pope, L. Romans and X. Shen, "The complete structure of W_∞", Phys. Lett. B236 (1990) 173; "W_∞ and the Racah-Wigner algebra", Nucl. Phys. B339 (1990) 191.

[17] M. Wakimoto, "Fock representations of the affine Lie algebra $A_1^{(1)}$", Commun. Math. Phys. 104 (1986) 605.

[18] B. Feigin and E. Frenkel, "Representations of affine Kac-Moody algebras and bosonization", in *Physics and Mathematics of Strings*, eds. L. Brink, D. Friedan and A. Polyakov, World Scientific, Singapore (1990).

[19] V. Kac and D. Kazhdan, "Structure of representations with highest weight of infinite dimensional algebras", Adv. Math. 34 (1979) 97.

[20] F. Ardalan, "Gauging of $SL(2,R)$ WZNW models and the Liouville field", Theor. Math. Phys. 98 (1994) 227.

[21] A. Yung, "World-sheet instantons in the 2d black hole", Int. J. Mod. Phys. A9 (1994) 591; "The broken phase of the topological sigma model", *ibid.* A10 (1995) 1553; "Instanton dynamics in the broken phase of the topological sigma model", *ibid.* A11 (1996) 951.

[22] J. Maldacena and H. Ooguri, "Strings in AdS_3 and the $SL(2,R)$ WZW model. I: The spectrum", J. Math. Phys. 42 (2001) 2929.